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Quantization of Christ–Lee Model Using the WKB Approximation

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The Hamilton–Jacobi formalism for constrained systems is applied to the Christ–Lee model. The equations of motion are obtained and the action integral is determined in the configuration space. This enables us to quantize the Christ–Lee model by using the WKB approximation.

KEY WORDS: Christ–Lee model; WKB approximation.

1. INTRODUCTION

The Christ–Lee model is viewed as a constrained system (singular system) (Christ and Lee, 1980; Costa *et al.*, 1985). The usual canonical quantization procedure may fail for constrained systems for several reasons: it may not be possible to eliminate some of velocities by a Legendre transform, or it may be that the Hamiltonian equations do not reproduce the desired dynamical equations. The basic ideas of the classical treatment and the quantization of constrained systems were presented a long time ago by Dirac (1964). He distinguishes between two types of constraints, first and second classes. Most physicists believe that this distinction is quite important not only in the classical theories but carries through in the quantum theories (Faddeev, 1969; Henneax and Teitelboim, 1992).

More recently, another powerful approach—the Hamilton–Jacobi formalism—has been developed for quantizing constrained systems (Güler, 1992; Rabei, 1996; Rabei and Güler, 1992). The equivalent Lagrangian method (Güler, 1989) is used to obtain the set of Hamilton–Jacobi partial differential equations (HJPDEs) for constrained systems. In this approach the distinction between first and second class constraints is not necessary. The equations of motion are written as total differential equations in many variables which require the investigation of integrability conditions. In previous work (Rabei, 1996) the link between the two approaches is studied. It is shown that the Hamilton–Jacobi approach is always in

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exact agreement with the Dirac approach. The integrability conditions are equivalent to the consistency conditions.

On the other hand, we have approached the problem of quantization of constrained systems in a somewhat interesting way (Rabei *et al.*, 2002). A general solution of the set of HJPDEs of these systems has been proposed, so that the Hamilton–Jacobi function in configuration space has been obtained. This formulation leads to another approach for solving mechanical problems for constrained systems in the same manner as for unconstrained systems. In addition, calculating this function enables us to quantize constrained systems using the WKB approximation (Rabei *et al.*, 2002).

In this paper we study the same procedure for the Christ–Lee model. It is organized as follows. In Section 2, the quantization of Christ–Lee model using Dirac method is reviewed. In Section 3, this model is quantized using the WKB approximation. The work closes with some concluding remarks in Section 4.

2. THE CHRIST–LEE MODEL IN STANDARD DIRAC QUANTIZATION METHOD

The Christ–Lee model is described by the singular Langrangian of the form,

$$L = \frac{1}{2}[\dot{r}^2 + r^2(\dot{\theta} - z)^2] - V(r). \tag{1.1}$$

The usual Hamiltonian is calculated as

$$H = \frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 + zp_\theta + V(r) \tag{1.2}$$

with the primary constraint

$$\Phi_1 = p_z \approx 0, \tag{1.3}$$

where r and θ are plane polar coordinates, z is another generalized coordinate, and $V(r)$ is a potential bounded from below (Costa *et al.*, 1985). Using the consistency conditions (Dirac, 1964), one finds the secondary constraints.

$$\Phi_2 = p_\theta \approx 0 \tag{1.4}$$

It is easy to check that there are no further constraints in the theory.

According to Dirac (1964) classification, the Hamiltonian (1.2) and the constraints (1.3) and (1.4) are first class. Thus, the total Hamiltonian reads as

$$H_T = H + \lambda p_z \tag{1.5}$$

This gives the following equations of motion:

$$\dot{r} = p_r, \tag{1.6}$$

$$\dot{\theta} = z, \tag{1.7}$$

$$\dot{z} = \lambda, \tag{1.8}$$

$$\dot{p}_r = -\frac{dV(r)}{dr}, \tag{1.9}$$

$$\dot{p}_\theta = 0, \tag{1.10}$$

$$\dot{p}_z = 0. \tag{1.11}$$

Equations (1.6) and (1.9) lead to the following second-order ordinary differential equation of the form

$$\ddot{r} + \frac{dV(r)}{dr} = 0. \tag{1.12}$$

Let us choose the potential to be harmonic $V(r) = \frac{1}{2}r^2$, then (1.12) has a solution of the form

$$r = A\cos t + B\sin t \tag{1.13}$$

and the corresponding momentum p_r reads as

$$p_r = -A\sin t + B\cos t. \tag{1.14}$$

On the other hand, to quantize the Christ–Lee model, the gauge-fixing conditions (Dayi, 1989) can be chosen as

$$\Phi_3 = z \approx 0, \tag{1.15}$$

$$\Phi_4 = \theta \approx 0. \tag{1.16}$$

These constraints make the system second class with $\Delta_{ab} = \{\Phi_a, \Phi_b\}$ where $a, b = 1, 2, 3, 4$. Thus one should obtain the Dirac brackets

$$\{r, p_r\}_{\mathrm{D}} = 1; \quad \{\theta, p_\theta\}_{\mathrm{D}} = 0; \quad \{z, p_z\}_{\mathrm{D}} = 0, \tag{1.17}$$

corresponding to the commutators:

$$[r, p_r] = i\hbar; \quad [\theta, p_\theta] = 0; \quad [z, p_z] = 0, \tag{1.18}$$

where the Dirac brackets for any functions $f(q, p), g(q, p)$ are defined as $\{f, g\}_{\mathrm{D}} = \{f, g\} - \{f, \Phi_a\}\Delta^{ab}\{\Phi_b, g\}$, with Δ^{ab} being the inverse of Δ_{ab}. Besides, the corresponding schrödinger equation reads,

$$-\frac{\hbar^2}{2}\frac{d^2}{dr^2}\Psi + V(r)\Psi = E\Psi. \tag{1.19}$$

3. THE WKB APPROXIMATION OF CHRIST–LEE MODEL

Following to Güler (1992) and Rabei and Güler (1992), the set of HJPDEs for constrained systems is written as

$$H_0' = p_0 + H_0;$$
$$H_\mu' = p_\mu + H_\mu. \qquad \mu = N - r + 1, \dots, N \tag{2.1}$$

Here $N - R$ is the rank of the Hessian matrix, H_0 is the usual Hamiltonian, H_μ' are the primary constraints, p_0 and p_μ are the canonical conjugate momenta to the time t and the coordinates q_μ respectively. In order words, $p_0 = \frac{\partial S}{\partial t}$ and $p_\mu = \frac{\partial S}{\partial q_\mu}$.

Following to Rabei *et al.* (2002), the general solution for Eq. (201) is proposed in the form

$$S(q_a, q_\mu, t) = f(t) + W_a(E_a, q_a) + f_\mu(q_\mu), \tag{2.2}$$

where $a = 1, 2, \dots, N - R$.

In the same reference (Rabei *et al.*, 2002), the equations of motion are obtained using the canonical transformations as

$$\lambda_a = \frac{\partial S}{\partial E_a}, \quad p_i = \frac{\partial S}{\partial q_i}, \qquad i = 1, 2, \dots, N \tag{2.3}$$

where λ_a are constants that can be determined from the initial conditions. These equations (2.3) can be solved to furnish q_a and the momenta p_i as

$$q_a = q_a(\lambda_a, E_a, q_\mu, t); \qquad p_i = p_i(\lambda_a, E_a, q_\mu, t). \tag{2.4}$$

The semiclassical expansion (WKB approximation) of the Hamilton–Jacobi function of constrained systems has been studied (Rabei *et al.*, 2002). This expansion leads to the following wave function

$$\Psi(q_a, q_\mu, t) = \left[\prod_{a=1}^{N-R} \Psi_{0a}(q_a) \exp\left(\frac{iS(q_a, q_\mu, t)}{\hbar}\right)\right], \tag{2.5}$$

where $\Psi_{0a}(q_a) = \frac{1}{\sqrt{p_a}}$. The above wave function (2.5) satisfies the conditions

$$H_0'\Psi = 0$$
$$H_\mu'\Psi = 0$$

in the semiclassical limit $\hbar \to 0$.

Now making use of Eq. (2.1), the set of HJPDEs to the Christ–Lee model can be obtained as

$$H_0' = p_0 + \frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 + zp_\theta + V(r) = 0,$$
$$H_z' = p_z = 0. \tag{2.6}$$

According to Rabei (1996), the integrability conditions which mean that the total differential of H'_0 and H'_z are equal to zero lead to new constraint

$$H'_\theta = p_\theta = 0 \tag{2.7}$$

Thus, the set of HJPDEs (201) can be rewritten as

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + V(r) = 0; \tag{2.8}$$

$$\frac{\partial S}{\partial z} = 0; \tag{2.9}$$

$$\frac{\partial S}{\partial \theta} = 0, \tag{2.10}$$

where $p_r = \frac{\partial S}{\partial r}$, $p_z = \frac{\partial S}{\partial z}$, and $p_\theta = \frac{\partial S}{\partial \theta}$.

According to Eq. (2.2) the function S has the following form

$$S = -Et + W(r, E) + f_1(z) + f_2(\theta). \tag{2.11}$$

Substituting in eqs. (2.8–10), we get

$$-E + \frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 + V(r) = 0, \tag{2.12}$$

$$\frac{\partial f_1}{\partial_z} = 0, \tag{2.13}$$

$$\frac{\partial f_2}{\partial_\theta} = 0. \tag{2.14}$$

Equations (2.13) and (2.14) give $f_1 = \text{constant}$ and $f_2 = \text{constant}$, while Eq. (2.12) gives

$$W = \int \sqrt{2(E - V(r))}\, dr. \tag{2.15}$$

Thus,

$$S = -Et + \int \sqrt{2(E - V(r))}\, dr + A, \tag{2.16}$$

where A is constant. The equations of motion read as

$$\lambda = \frac{\partial S}{\partial E} = -t + \int \frac{dr}{\sqrt{2(E - V(r))}},$$

$$p_r = \frac{\partial S}{\partial r} = \sqrt{2(E - V(r))}. \tag{2.17}$$

Taking $V(r) = \frac{1}{2}r^2$, and inserting in Eq. (2.17), one finds

$$r = \sqrt{2E}\,\sin(t + \lambda), \tag{2.18}$$

$$p_r = \sqrt{2E}\,\cos(t + \lambda). \tag{2.19}$$

Choosing $A = \sqrt{2E}\sin\lambda$ and $B = \sqrt{2E}\cos\lambda$, one observes that Eqs. (2.18) and (2.19) are in exact agreement with Eqs. (1.13) and (1.14).

We are now in a position to quantize our system. Making use of Eq. (2.5) the wave function for the Christ–Lee model can be written as

$$\Psi(r, t) = \frac{1}{[2(E - V(r))]^{1/4}}\,\exp\left(\frac{-iE}{\hbar}\right)\exp\left(\frac{i}{\hbar}\int\sqrt{2(E - V(r))}\,dr\right). \tag{2.20}$$

The above wave function (2.20) satisfies the Schrödinger equation (1.19) in the semiclassical limit $\hbar \to 0$.

4. CONCLUSION REMARKS

In this work the proposed general method of our previous work (Rabei *et al.*, 2002) for determining the Hamilton–Jacobi function of constrained systems is applied to the Christ–Lee model. This function is used to construct a suitable wave function for the model.

Following Henneax *et al.* (1990) the number of physical degrees of freedom of the Christ–Lee model is one. In this formalism we have shown that the action integral and the wave function Ψ are obtained in terms of the generalized coordinate r and the time t. In other words there is only one generalized coordinate r and the corresponding generalized momentum p_r. This is in exact agreement with Henneax *et al.* (1990) and Baleanu and Güler (2000).

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